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An Introduction to Rota-Baxter Algebra

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An Introduction to Rota-Baxter Algebra

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Surveys of Modern Mathematics

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Preface

This is an introduction to Rota-Baxter algebra that aims at presenting basic results of Rota-Baxter algebra with minimal background requirement for the readers.

A Rota-Baxter algebra (first known as a Baxter algebra) is an associative algebra equipped with a linear operator that generalizes the integral operator in analysis. Thus the concept of a Rota-Baxter algebra could have been introduced as an algebraic abstraction of the integral analysis, as in the case of a differential algebra which was studied initiatively by J. F. Ritt in the first third of the twentieth century as an algebraic study of differential equations. Instead Rota-Baxter algebra originated from the 1960 paper [24] of G. Baxter based on his probability study to understand Spitzer's identity in fluctuation theory. Soon afterwards, this concept attracted the attention of well-known mathematicians such as F. V. Atkinson, P. Cartier, and especially G.-C. Rota whose fundamental papers around 1970 brought the subject into the areas of algebra and combinatorics and whose survey articles in the 1990s helped its renascence several years later. While the development of Rota-Baxter (associative) algebra entered a dormant period from the late 1970s to the late 1990s, the corresponding concept for Lie algebra was studied independently by the mathematical physicists, under the name of (the operator form of) the classical Yang-Baxter equation (CYBE), named after the well-known physicists C. N. Yang and R. Baxter.

By coincidence, several discoveries on Rota-Baxter algebra were published just around the turn of the century. After the work of Winkel [173] on Baxter sequences in 1998, Rota-Baxter algebras were found their application in the profound work of Connes and Kreimer [42, 43] in 2000 on the renormalization of perturbative quantum field theory.

Another connection with mathematical physics was also established in year 2000 in the paper [2] of Aguiar that related a Rota-Baxter algebra of weight zero to the associative analog of the classical Yang-Baxter equation [3]. Aguiar further showed that a Rota-Baxter algebra of weight zero naturally carries a structure of a dendriform algebra which had been introduced by Loday [131] in the study of K-theory. Incidentally, a basic example of dendriform algebras is the shuffle product algebra that had been suggested to Loday [134] by Rota and would be shown again coming from Rota-Baxter algebras [53].

Still in year 2000, Keigher and the author, following a suggestion [159] of Rota in an early paper [91], realized free Rota-Baxter algebras as a generalization of the shuffle product [92], called the mixable shuffle product. This did not only facilitated further studies [93, 83, 85] of Rota-Baxter algebras, but also suggested links between these algebras to the many areas related to the shuffle product.

In the same year, a paper of Hoffman [112] introduced another generalization of the shuffle product, called the quasi-shuffle product which had played a fundamental role in the theory of multiple zeta values [111] and quantum field theory [123], and which would turn out to be the same as the mixable shuffle product [53].

Since then, many articles on Rota-Baxter algebras have been completed, in connection with quantum field theory, operads, Hopf algebras, commutative algebra, combinatorics and number theory. An updated list of articles on Rota-Baxter algebra and related areas is maintained at the web site http://andromeda.rutgers.edu/~liguo/rba.html.

The broad connections of Rota-Baxter algebra with many areas in mathematics and mathematical physics are remarkable. Yet the theoretical study of Rota-Baxter algebra is still in its early stage of development in comparison with the rich theory of differential algebra, thus holding the great potential for future researches. These features should make the study of Rota-Baxter algebra attractive to readers with the diverse background. Thus there is a practical need for a monograph on Rota-Baxter algebra to organize the related results spread out in the literature, in addition to the existing survey articles on various aspects of Rota-Baxter algebra. It is the purpose of this monograph to put together the basic properties and constructions of Rota-Baxter algebra as an introduction to two groups of readers who would like to get an overview of Rota-Baxter algebra and who would like to go further to work in this direction. To serve the first group of readers, the main body of the book treats in detail "classical" results that have appeared for several years. The prerequisite is a basic course in abstract algebra either on the undergraduate or the graduate level. Other background will be introduced if need. To serve the second group of readers, a summary has been included at the end of each chapter to briefly discuss more recent developments with the hope to familiarize the readers with some open problems and some active areas of the current research. These summaries are necessarily incomplete, sketchy and assume further background on the part of the readers. A more comprehensive and detailed discussion will have to wait for a future treatise.

The book is divided into three parts, covering three main aspects of Rota-

Baxter algebra, more or less from the author's viewpoint [90]. Each part consists of two chapters. After the basic Section 1.1, the reader can go on with any of the three parts. There is even some flexibility within each part. For example, Chapter 4 is mostly independent of Chapter 3. Overall enough cross references have been provided so that the reader can choose a section and follow the links to refer back to the earlier sections of the book when needed. A Leitfaden is provided for ease of the navigation.

The first part is on the operator aspect of Rota-Baxter algebra that emphasizes the properties of the Rota-Baxter operator, including Spitzer's identity, Atkinson's factorization and its relation to the algebraic Birkhoff decomposition in the Hopf algebra approach of Connes and Kreimer on renormalization of quantum field theory. The concepts, examples and basic properties of Rota-Baxter algebra are introduced in chapter 1. But the main focus of this chapter is to develop various formulations of Spitzer's identity and the related Atkinson's decompositions. Chapter 2 applies Chapter 1 to the special setting where the Rota-Baxter algebra is the convolution algebra of the linear maps from a Hopf algebra to a commutative Rota-Baxter algebra, after the background on connected bialgebras is discussed. This chapter ends with the algebraic Birkhoff decomposition that is applied to renormalization in both physics and mathematics contexts.

The more algebra-oriented readers might find the second part more interesting, since it follows a more traditional treatment of an algebra structure, namely on explicit constructions of the free objects. Because of the rich structure of a Rota-Baxter algebra, we construct free Rota-Baxter algebras in various contexts: commutative versus noncommutative, unitary versus nonunitary, generated by set versus algebras, as well as various representations of the linear bases. Chapter 3 is devoted to the constructions of free commutative Rota-Baxter algebras and their applications to Stirling numbers and partitions. By giving the construction of free commutative Rota-Baxter algebras in terms of the equivalent products of mixable shuffle, quasi-shuffle, stuffle and Delannoy paths, we underline the broad interests from this purely algebraic construction. The original construction of free commutative Rota-Baxter algebras of Rota and Cartier are also discussed, in connection with Spitzer's identity and Waring's formula on symmetric functions. Chapter 4 studies free (noncommutative) Rota-Baxter algebras. We construct the free Rota-Baxter algebra on a set and on an algebra. Further we give the constructions in terms of bracketed words and rooted trees, and in the categories of unitary and nonunitary Rota-Baxter algebras. As an application, we establish the existence and uniqueness of the unitarization of Rota-Baxter algebras.

The third part of the book is on the operad aspect of Rota-Baxter algebra, exploring the connection between Rota-Baxter algebra and operad objects, such as the dendriform algebra. In Chapter 5, we compare free commutative dendriform algebras and free commutative tridendriform algebras with free commutative Rota-Baxter algebras. Then in the non-necessary commutative context, we consider the adjoint functor and universal enveloping algebras related to the functor from Rota-Baxter algebras to (tri-)dendriform algebras. In Chapter 6, we work in the broader context of binary quadratic nonsymmetric operads and consider the relationship between Rota-Baxter operads on such operads and the black square product.

It is my great pleasure to thank all my coauthors, from A(ndrews) to Z(hang), whose collaborations have helped to expose my interests to a wide range of areas in mathematics and physics, and therefore made the writing of this book possible. Particularly, most results in this book came from joint works with them. This book is based on my lectures for graduate courses given at Rutgers University at Newark, Lanzhou University and Zhejiang University. I thank the participants from these courses for their interests. Special appreciation goes to my colleagues and students who have helped to improve the previous drafts, including M. Aggarval, C. Bai, K. Ebrahimi-Fard, X. Gao, H. Kapadia, W. Keigher, P. Lei, F. Li, Z. Lin, J.-L. Loday, Y. Luo, J. Pei, Z. Shi, W. Y. Sit, L. Sun, L. Thrall, B. Vallette, C. Valverde, J. Wang, R. Zhang, Y. Zhang, S. Zheng and C. Zhou, S. Zhou. Thanks also go to my editors Li Zhen Ji and Liping Wang for their encouragement and patience. This book was writing while the author was supported in part by NSF grants DMS-0505643 and DMS-1001855. Finally, I dedicate this book to my family: my parents, my sister, my wife, and my children Audrey and Kevin for their love and support.

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An Introduction to Rota-B has been added to your Cart. Add a gift receipt with prices hidden. Other Sellers on Amazon.Â Pages with related products. See and discover other items: modern algebra. There's a problem loading this menu right now. Learn more about Amazon Prime. 1. INTRODUCTION. In this paper we generalize the well-known construction of shuffle product algebras by using mixable shuffles, and we prove that any free Baxter algebra is isomorphic to a mixable shuffle product algebra. This gives an explicit construction of the free Baxter algebra, extending the work of Rota [15] and Cartier [2]. In an important paper published in 1958, Ree [12] constructed algebras in which the product is expressed in terms of shuffles.Â It was Rota who pointed out to us the earlier work on free Baxter algebras and suggested that we extend our shuffle product description of free integration algebras to Baxter algebras. This is carried out in this paper, by making use of a modified shuffle product, called the mixable shuffle product. In mathematics, a Rotaâ€“Baxter algebra is an associative algebra, together with a particular linear map R which satisfies the Rotaâ€“Baxter identity. It appeared first in the work of the American mathematician Glen E. Baxter in the realm of probability theory. Baxter's work was further explored from different angles by Gian-Carlo Rota, Pierre Cartier, and Frederic V. Atkinson, among others. Baxterâ€™s derivation of this identity that later bore his name emanated from some of the fundamental results of the Rota-Baxter algebras (and categories) of arbitrary weights are discussed in section 4 below. Clearly every dierential category is a Leibniz category. That the converse is not true is a demonstration of the utility of the Leibniz category denition, and is what we will spend much of the rst part of this paper showing. The main Leibniz example for us will be the free dierential algebra. Of course, the same statements hold with respect to integral and Rota-Baxter categories, and this will be likewise explored below. The main Rota-Baxter example for us will be the free Rota-Baxter algebra.