**HOLOMORPHIC CURVES IN SYMPLECTIC AND CONTACT GEOMETRY**
**HU BERLIN – WINTER SEMESTER 2009-10**

## Instructor

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## Description

Pseudoholomorphic curves, also called *J–holomorphic* or simply *holomorphic curves* for short, are solutions to a first order elliptic PDE that generalizes the Cauchy-Riemann equation for Riemann surfaces mapping into "almost" complex manifolds. Gromov showed in his famous 1985 paper that a natural setting in which to study these objects is furnished by *symplectic manifolds*: these are even-dimensional manifolds that look locally like the Hamiltonian "phase space" of classical mechanics. Since symplectic manifolds are all locally the same (quite unlike the situation in Riemannian geometry), most interesting symplectic questions are of a more *global* nature, and on this subject very little was known until Gromov's work revealed that the compactness properties of holomorphic curves encode global symplectic invariants, implying for instance the celebrated "non-squeezing" theorem. Some years later, holomorphic curves also found application in *contact geometry*, the odd-dimensional analogue. Contact manifolds arise naturally as regular level surfaces of Hamiltonians in symplectic manifolds, or as boundary components of *symplectic cobordisms*: the latter furnish a natural setting in which to study *punctured* holomorphic curves with cylindrical ends, producing a more TQFT-style picture that yields invariants in contact topology.

In *dimension four*, the theory of holomorphic curves has a distinctive flavor due to intersection theory: the algebraic intersections of holomorphic curves are always positive, and are thus easy to control. The resulting interaction between topology and analysis produces much stronger compactness results, with beautiful applications, e.g. in some situations, the existence of a single holomorphic curve implies a geometric decomposition of its ambient manifold, essentially determining that manifold up to symplectomorphism.

The goal of this course is to develop enough of the basic theory of holomorphic curves to understand some of its classic applications, with emphasis on the 4–dimensional case. The first two thirds of the course will focus on closed holomorphic curves in closed symplectic manifolds, culminating in the proof of Gromov's non-squeezing theorem and McDuff's classification of rational and ruled symplectic 4–manifolds using embedded holomorphic spheres. The last third will then extend these ideas to the more general context of punctured holomorphic curves in symplectic cobordisms, with applications to 3–dimensional contact topology, including obstructions to symplectic filling and the Weinstein conjecture. Along the way, we'll discuss a number of important technical details, including the implicit function theorem in Banach manifolds, Fredholm theory and transversality, "bubbling off" analysis, positivity of intersections and the adjunction formula. Fundamental concepts from symplectic and contact geometry (e.g. Hamiltonian vector fields, Darboux's theorem and Moser deformation arguments, Maslov index) will also be introduced as needed.

## Literature

The main text for the course will be an extensive set of lecture notes by the instructor. Aside from these, the book of McDuff and Salamon [MS04] is essential for anyone serious about the subject; most of the topics in the first 2/3 of the course are treated there, though not always in exactly the same way that we'll treat them in the lecture. Another book containing much of the same material is [94]. A surprisingly large proportion of it all originates in Gromov's paper [Gro85], which is difficult for beginners but worth rereading once every few years. Additional references for various specific topics will be recommended as needed; several are cited in the syllabus below.

## Prerequisites

- Differential Geometry (manifolds, vector fields, vector bundles, differential forms and Stokes theorem)
- Functional Analysis (linear operators on Banach spaces, spectral theory, familiarity with Sobolev spaces)
- Algebraic Topology (fundamental group, homology and cohomology of manifolds, Poincaré duality, first Chern class)

## Syllabus

The following is meant as a week-by-week breakdown of the topics to be covered. It is only tentative.

1. Introduction, some basics of symplectic and contact topology [MS98]
2. Nonlinear Cauchy-Riemann operators and linearizations, linear elliptic regularity, unique continuation and intersections of $J$–holomorphic curves
3. Banach manifolds and bundles [Lan93, Lan99], local existence and regularity of $J$–holomorphic curves
4. Fredholm theory, Riemann-Roch, Teichmüller spaces
5. Sard-Smale theorem, transversality for somewhere injective $J$–holomorphic curves
6. Sketch of Deligne-Mumford compactification [SS92], bubbling off analysis
7. Gromov compactness, non-squeezing theorem
8. Special properties in dimension 4: automatic transversality, positivity of intersections, adjunction formula
9. Exceptional spheres and blowups, classifying rational and ruled symplectic 4–manifolds [McD90]
10. Contact manifolds, symplectizations, symplectic cobordisms, stable Hamiltonian structures, punctured holomorphic curves [EGH00]
11. Fredholm and compactness theory in symplectic cobordisms [Wena, BEH+03]
12. Automatic transversality and intersection theory of punctured holomorphic curves [Wena, Sie08, Sie]
13. Applications to symplectic filling of contact 3–manifolds, the Weinstein conjecture [Wenb]
14. Additional topics as time permits

# References


[94] *Holomorphic curves in symplectic geometry*, Progress in Mathematics, vol. 117, Birkhäuser Verlag, Basel, 1994. Edited by Michèle Audin and Jacques Lafontaine.

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[EGH00] Y. Eliashberg, A. Givental, and H. Hofer, *Introduction to symplectic field theory*, Geom. Funct. Anal., Special Volume (2000), 560–673.

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[Lan93] S. Lang, *Real and functional analysis*, 3rd ed., Springer-Verlag, New York, 1993.

[Lan99] ———, *Fundamentals of differential geometry*, Springer-Verlag, New York, 1999.

[McD90] D. McDuff, *The structure of rational and ruled symplectic* 4*-manifolds*, J. Amer. Math. Soc. **3** (1990), no. 3, 679–712.

[MS98] D. McDuff and D. Salamon, *Introduction to symplectic topology*, The Clarendon Press Oxford University Press, New York, 1998.

[MS04] ———, *$J$-holomorphic curves and symplectic topology*, American Mathematical Society, Providence, RI, 2004.

[SS92] M. Seppälä and T. Sorvali, *Geometry of Riemann surfaces and Teichmüller spaces*, North-Holland Mathematics Studies, vol. 169, North-Holland Publishing Co., Amsterdam, 1992.

[Sie08] R. Siefring, *Relative asymptotic behavior of pseudoholomorphic half-cylinders*, Comm. Pure Appl. Math. **61** (2008), no. 12, 1631–1684.

[Sie] ———, *Intersection theory of punctured pseudoholomorphic curves*. Preprint arXiv:0907.0470.

[Wena] C. Wendl, *Automatic transversality and orbifolds of punctured holomorphic curves in dimension four*. To appear in Comment. Math. Helv., Preprint arXiv:0802.3842.

[Wenb] ———, *Strongly fillable contact manifolds and $J$–holomorphic foliations*. To appear in Duke Math. J., Preprint arXiv:0806.3193.

Start by marking â€œHolomorphic Curves in Symplectic Geometryâ€ as Want to Read: Want to Read savingâ€¦ Want to Read.Â  Duval was not able to write up his lectures, so that genuine complex analysis will not appear in the book, although it is a very current tool in symplectic and contact geometry (and conversely). Hamiltonian systems and variational methods were the subject of some of Sikorav's talks, which he also was not able to write up. On the other hand, F. Labourie, who could not be at the school, wrote a chapter on pseudo-holomorphic curves in Riemannian geometry. ...more. Get A Copy. Amazon. In mathematics, specifically in topology and geometry, a pseudoholomorphic curve (or J-holomorphic curve) is a smooth map from a Riemann surface into an almost complex manifold that satisfies the Cauchyâ€“Riemann equation. Introduced in 1985 by Mikhail Gromov, pseudoholomorphic curves have since revolutionized the study of symplectic manifolds. In particular, they lead to the Gromovâ€“Witten invariants and Floer homology, and play a prominent role in string theory. A parametrized (pseudo holomorphic) J-curve in an almost complex manifold (IS, J) is a holomorphic map of a Riemann surface into Is, say f : (S, J3 ~(V, J). The image C=f(S)C V is called a (non-parametrized) J-curve in V. A curve C C V is called closed if it can be (holomorphically !) parametrized by a closed surface S. We call C regular if there is a parametrization f : S ~ V which is a smooth proper embedding. A curve is called rational if one can choose S diffeomorphic to the sphere S 2.Â  @article{Gromov1985PseudoHC, title={Pseudo holomorphic curves in symplectic manifolds}, author={M. Gromov}, journal={Inventiones mathematicae}, year={1985}, volume={82}, pages={307-347} }. M. Gromov. Published 1985.