(*Relativizations of the $\mathcal{P} =? \mathcal{NP}$ question*, JSL LI 1061), which state that there are oracles α, β relativizing P and NP such that $\mathrm{P}^\alpha = \mathrm{NP}^\alpha$ and $\mathrm{P}^\beta \neq \mathrm{NP}^\beta$, cannot be applied to his setting.

I have one general concern: If we consider a certain entity of Gödel numbers T (any kind of Gödelization) of closed terms in the language of BA and another entity T_x of Gödel numbers of terms in one fixed variable x, and if we further consider an evaluation function $val : T \to \mathbb{N}$ and a substitution function $sub : T_x \times \mathbb{N} \to T$, then faithfulness of the arithmetization implies that all polynomial growth rates can be represented in the system (T, T_x, val, sub). For proving Gödel's diagonalization lemma we need the substitution function sub to have a polynomial growth rate, and for showing the correctness of derivations in BA by the evaluation strategy we need the evaluation function val to have a polynomial growth rate. But this is impossible, because then $f(a,b) := val(sub(a,b))$ has polynomial growth rate, i.e., $|f(a,b)| \leqq |\max(a,b)|^c$ for some $c \in \mathbb{N}$. By faithfulness there is some $t_{c+1} \in T_x$ such that $|m|^{c+1} \leqq |val(sub(t_{c+1},m))| \leqq |\max(t_{c+1},m)|^c = |m|^c$ for m big enough, a contradiction. I think this indicates that in order to obtain a separation via Gödel sentences we need further ideas that differ strongly from the evaluation strategy.

The first reviewed paper summarizes all necessary notions and results on strictly i-normal proofs and partial truth definitions for them. Let $\mathrm{Prf}^i(w, \ulcorner \varphi \urcorner)$ denote "w is the Gödel number of a strictly i-normal proof of φ," and let $|x|_k$ denote the k-times iterated "integer-logarithm" $|\cdot|$ applied to x. Takeuti shows that BA can prove weak consistency statements like $\forall w \neg \mathrm{Prf}^i(|w|, \ulcorner \to \urcorner)$ (the empty sequent "$\to$" representing contradiction) and weak forms of reflection like $\exists w\, \mathrm{Prf}^i(|w|_2, \ulcorner \forall x\, \varphi(x) \urcorner) \to \forall x\, \varphi(|x|)$ for simple φ. Of course, Gödel's second incompleteness theorem can only be proved for the "usual" consistency statement $S_2^i \nvdash \forall w \neg \mathrm{Prf}^i(w, \ulcorner \to \urcorner)$. Takeuti defines Gödel sentences φ_k^i satisfying $S_2^i \vdash \varphi_k^i \leftrightarrow \forall x \neg \mathrm{Prf}^i(|x|_k, \ulcorner \varphi_k^i \urcorner)$. They have similar properties to the above-described variations of consistency. By Gödel's diagonalization lemma, Takeuti obtains $S_2^i \nvdash \varphi_k^i$ for all k, but on the other hand $S_2^{i+1} \vdash \forall x \neg \mathrm{Prf}^i(|x|_{k+1}, \ulcorner \varphi_k^i \urcorner)$ for $k \geqq 2$. He formulates two conjectures in the above-described form that would imply $\mathrm{P} \neq \mathrm{NP}$.

The second reviewed paper extends this line of research and contrasts it with Gödel sentences based on the usual (unrestricted) derivability. That is, let $\mathrm{PRF}^i(w, \ulcorner \varphi \urcorner)$ denote the formalized notion of "w is the Gödel number of an (unrestricted) S_2^i-proof of φ," and define Gödel sentences Φ_k^i satisfying $S_2^1 \vdash \Phi_k^i \leftrightarrow \forall x \neg \mathrm{PRF}^i(|x|_k, \ulcorner \Phi_k^i \urcorner)$. Takeuti compares pairs $(\mathrm{PRF}^i, \Phi_k^i)$ with $(\mathrm{Prf}^i, \varphi_k^i)$, obtaining results like $I\Delta_0 + \exp \nvdash \forall x \neg \mathrm{PRF}^i(|x|_{k+1}, \ulcorner \Phi_k^i \urcorner)$ for all k, in contrast to the above-obtained results for $(\mathrm{Prf}^i, \varphi_k^i)$. He further establishes relationships between different Gödel sentences, and between Gödel sentences and weak consistency statements.

The reviewed papers are top-level research papers in the field of applications of Gödel sentences to the separation problem of BA. This implies that for the study of them one needs some background in BA (at least parts of the Buss book). Like all papers of Takeuti that I have read, I found the ones under review inspiring, showing new ideas to the separation problem of BA.

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ANDREAS WEIERMANN. *How is it that infinitary methods can be applied to finitary mathematics? Gödel's T: a case study*. **The journal of symbolic logic**, vol. 63 (1998), pp. 1348–1370.

The paper under review is concerned with Gödel's system T of primitive recursive functionals of finite type, which is sufficient for providing a functional interpretation of first-order Peano arithmetic PA. The main technical achievement of the paper is a novel proof of the

strong normalization of T by defining a new assignment $[\ ]_0 : T \to \omega$ which decreases under one-step reduction $\rhd^1$ of terms in T, i.e., $t \rhd^1 t'$ entails $[t']_0 < [t]_0$. Moreover, $[\ ]_0$ is an ε_0-recursive function. As a byproduct, optimal bounds for the natural fragments T_n of T are obtained.

As the title of the paper suggests, this new analysis of Gödel's T has to be seen as a specific case study of a more general program, which has its roots in infinitary proof theory. There the general concept of *miniaturization* and *concretion* of certain (large) cardinals has long been central in the development of ordinal notation systems. Further, as has been shown by Weiermann, it is often possible to miniaturize or project down the ordinal analysis of a (strong) system in order to obtain a perspicuous proof-theoretic analysis of a corresponding weaker system. For example, he has shown in *How to characterize provably total functions by local predicativity* (**The journal of symbolic logic**, vol. 61 (1996), pp. 52–69) how to pin down the standard local predicativity treatment (due to Pohlers) of the theory of one inductive definition ID_1 to a technically smooth analysis of Peano arithmetic PA. In this miniaturization, the collapsing function $D : \varepsilon_{\Omega+1} \to \Omega$ is replaced by the miniaturized collapsing function $\psi : \varepsilon_0 \to \omega$. The treatment of PA thus obtained indeed also produces optimal bounds for the fragments $I\Sigma_{n+1}$ of PA.

The treatment of Gödel's T in the paper under review can be seen as a miniaturization of Howard's analysis of bar recursion of type zero and is inspired by the local predicativity approach to pure proof theory. In particular, the definition of the assignment $[\ ]_0$ mentioned above makes crucial use of the miniaturized collapsing function ψ. Whereas previous treatments of the subsystems T_n of T (in which the recursors have type level less than or equal to $n + 2$) used the ordinal bound ω_{n+3} (e.g. in Weiermann's paper *A proof of strongly uniform termination for Gödel's T by methods from local predicativity*, **Archive for mathematical logic**, vol. 36 (1997), pp. 445–460—a precursor of the article under review), the present paper yields the optimal strong normalization bound ω_{n+2} for T_n.

This is a very important paper both from the technical as well as from the conceptual point of view. In a long and sparkling introduction, the author outlines his vision of how to apply a certain kind of infinitary methods to questions of finitary mathematics, and he discusses exciting connections of his results with term rewriting theory, hierarchy theory, and computational complexity. The paper is largely self-contained and, due to its extensive motivation and conceptual discussion, it should be accessible to a wide readership.

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ULRICH KOHLENBACH. *Relative constructivity*. **The journal of symbolic logic**, vol. 63 (1998), pp. 1218–1238.

In this article, the author continues his investigations of the rate of growth of functions definable in certain subsystems of analysis in finite types. In previous papers (e.g. *Mathematically strong subsystems of analysis with low rate of growth of provably recursive functionals*, BSL VII 280) the classical systems $G_n A^\omega$ were analyzed and proof-theoretic tools such as monotone functional interpretation (introduced by the author) were used. In the paper under review, Kohlenbach studies the intuitionistic versions $G_n A_i^\omega$ of these systems and analyzes them via a new monotone realizability interpretation. He shows that the addition of certain strong non-constructive and even classically refutable analytical and logical principles has no impact on the rate of growth of definable functions.

The main result of the paper is summarized in the following.

Main theorem. Let E-$G_n A_i^\omega$ be the system $G_n A_i^\omega$ plus extensionality axioms in all finite types, the full axiom of choice AC in all finite types, and the independence of premiss principle IP$_\neg$ for negated formulas. Furthermore, let $\mathcal{A}$ consist of the following (classically

A. Weiermann, How is it that infinitary methods can be applied to finitary mathematics? reviewed by Thomas Strahm, page 435. U. Kohlenbach, Relative constructivity, reviewed by Ulrich Berger, page 436. Three papers on proof theory by W. Buchholz and by S. Tupailo, reviewed by Toshiyasu Arai, page 437. Â T. Arai, Consistency proof via pointwise induction, reviewed by Andreas Weiermann, page 536. M. Rathjen, The superjump in Martin-LÃ¶f type theory, reviewed by Michael MÃ¶llerfeld, page 538. S. Feferman, Computation on abstract data types, reviewed by Jeffery Zucker, page 538. S. Shelah and H. Woodin, Large cardinals imply that every reasonably definable set of reals is Lebesgue measurable, reviewed by Joan Bagaria, page 543. An infinitary logic is a logic that allows infinitely long statements and/or infinitely long proofs. Some infinitary logics may have different properties from those of standard first-order logic. In particular, infinitary logics may fail to be compact or complete. Notions of compactness and completeness that are equivalent in finitary logic sometimes are not so in infinitary logics. Therefore for infinitary logics, notions of strong compactness and strong completeness are defined. This article GÃ¶delâ€™s two incompleteness theorems are among the most important results in modern logic, and have deep implications for various issues. They concern the limits of provability in formal axiomatic theories. Â There have been attempts to apply the results also in other areas of philosophy such as the philosophy of mind, but these attempted applications are more controversial. Â GÃ¶delâ€™s theorem does not merely claim that such statements exist: the method of GÃ¶delâ€™s proof explicitly produces a particular sentence that is neither provable nor refutable in F; the â€œundecidableâ€ statement can be found mechanically from a specification of F. The sentence in question is a relatively simple statement of number theory, a purely universal arithmetical sentence. GÃ¶delâ€™s T: A Case Study. The Journal of Symbolic Logic 63(4), 1348â€"1370 (1998)MathSciNetCrossRefzbMATHGoogle Scholar. Copyright information. Â© Springer-Verlag Berlin Heidelberg 2009. Authors and Affiliations. Gunnar Wilken. 1. Andreas Weiermann. Â Wilken G., Weiermann A. (2009) Complexity of GÃ¶delâ€™s T in Î»-Formulation. In: Curien PL. (eds) Typed Lambda Calculi and Applications. TLCA 2009. Lecture Notes in Computer Science, vol 5608. Springer, Berlin, Heidelberg. https://doi.org/10.1007/978-3-642-02273-9_28. DOI https://doi.org/10.1007/978-3-642-02273-9_28. Publisher Name Springer, Berlin, Heidelberg. GÃ¶del's T: a case study. The journal of symbolic logic, vol. 63 (1998), pp. 1348â€"1370. Volume 8, Issue 3. Â Your email address will be used in order to notify you when your comment has been reviewed by the moderator and in case the author(s) of the article or the moderator need to contact you directly. Please enter a valid email address. Occupation Please enter your occupation.