Noncommutative Geometry is an area of mathematics which has been dominated by the work of Alain Connes in the last 20-25 years. The basic idea is that instead of point sets (e.g. manifolds) one studies the coordinate ring of (smooth) functions. This point of view has been around in algebraic geometry for decades but it was Connes' who showed that also manifolds and index theory can be understood from this perspective. Examples of 'noncommutative spaces', where now the coordinate ring is a noncommutative algebra, are abundant and Noncommutative Geometry is an area of active current research.

The purpose of this seminar is modest. We want to study some of the basic material of noncommutative geometry, like cyclic (co)homology, Fredholm modules, the noncommutative analogue of the classical Chern character and the Hochschild–Kostant–Rosenberg–Connes Theorem.

For a first reading we will use the excellent survey by Higson [11]. Since I also hope to attract international graduate students the seminar is going to be held in English.

**When and Where:** Flexible. Options: Do, 8-10, SR B or Mo, 14-16, SR A

**First meeting:** Mo, 02.04.2007, 14:15, Beringstr. 4, Seminarraum A.

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**Talks**

1. **The Gelfand–Naimark Theorem and noncommutative topology.**

   Material: [2], in particular the introductory section on (commutative) $K$–theory, [15, Sec. 4.2], [10, Chap. I]
Explain the content of the Gelfand–Naimark theorem and the basic dictionary of non–commutative topology. This dictionary expresses basic topological notions in terms of the algebra of continuous functions. Review the main results of $K$–theory for $C^*$–algebras.

2. The trace and the Schatten ideals.

Material: [15, Sec. 3.4], [16]

Follow the exposition in [15, Sec. 3.4] and introduce the trace and trace class operators on a Hilbert space. The trace is a positive linear functional, taking possibly the value $+\infty$, on the cone of nonnegative operators. The theory of the trace in many respects resembles measure theory. This should be emphasized and therefore also the so–called Schatten $p$–ideals [15, E. 3.4.2-E.3.4.4] should be discussed. Another crucial result is the fact that the map $(S, T) \mapsto \text{tr}(ST)$, where $S$ is bounded and $T$ is trace class, implements the duality between the Banach space of trace class operators and the Banach space of bounded linear operators. This generalizes the well–known duality between $L^1$ and $L^\infty$ in measure theory.

In addition to [15] one should briefly address the problem whether for a trace class operator $T$ the sequence of eigenvalues is summable and $\text{tr}(T)$ equals the sum of the eigenvalues. For self–adjoint operators this is easy to see and it should be presented. The general case, the so–called Lidskii–Theorem, is more difficult. Consult [16] and give at least a couple of comments.

3. Fredholm modules and the index pairing.

Material: [11, Sec. 2.1], [3, Appendix], [4, Appendix IV.A], [12, Chap. 8]

The material of [11, 2.1] should be covered. However, some more background on the role of Fredholm modules as cycles in $K$–homology should be given. $K$–homology is the dual theory to $K$–theory and there is a natural bilinear pairing

$$\text{index} : K_j(A) \times K^j(A) \rightarrow \mathbb{Z}.$$ 

Since Fredholm modules are the cycles in $K$–homology, every Fredholm module induces naturally a map $K_j(A) \rightarrow \mathbb{C}$. This map should be explained in detail in both the even and in the odd case ([11, Sec. 2.1], [4, Prop. IV.2]).

Explain the difference between algebraic $K_j$ and topological $K_j$ for $j = 0, 1$.

Then finitely summable Fredholm modules should be discussed. The following simple but important Lemma should be proved:
Proposition. Let $H$ be a Hilbert space and $T \in \mathcal{B}(H)$ a Fredholm operator. Furthermore let $S \in \mathcal{B}(H)$ be a parametrix of $T$ such that $(I - ST)^k, (I - TS)^k$ are trace class for some integer $k \geq 1$. Then

$$\text{ind}(T) = \text{tr}((I - ST)^k) - \text{tr}((I - TS)^k).$$

4. The character of a finitely summable Fredholm module.

Material: [11, Sec. 2.2], [4, Sec. IV.1],

Present [11, Sec. 2.2] giving full proofs of Theorems 2.10 and Theorem 2.11. For more details consult [4, Sec. IV.1]. Try to give a proof of Theorem 2.7 without becoming too technical about cyclic cohomology. Consult [3].

5. Hochschild (co)homology.

Material: [14, Chap I], [11, Sec. 2.3], [7], [18]

Concentrate on Sections 1.0, 1.1, 1.5 and 1.6 of Loday’s book. You may use the framework of simplicial modules but you should avoid a too high level of abstraction. The main examples of simplicial modules are the Hochschild complex and co-complex.

The bar resolution Prop. 1.1.12 and the normalized Hochschild complex (1.1.14) should be discussed thoroughly. Loday uses a spectral sequence argument. Also for later purposes it might be helpful to give a concise presentation of the spectral sequence of a double complex [14, Appendix D].

It will help to demystify the spectral sequence argument if you give a short direct proof of the acyclicity of the normalized Hochschild complex using the displayed formula in the proof of [14, Lemma 1.6.6].

6./7. Cyclic (co)homology I and II.

Material: [14, Chap II], [11, Sec. 2.3], [4, Sec. III.1, III.3], [7], [18]

Follow Loday’s book and present the various complexes which calculate cyclic (co)homology (cyclic bicomplex, Connes’ complex, Connes’ bB–bicomplex). Connes’ SBI–sequence and the periodicity operator as well as period cyclic (co)homology should be covered, too. Note that in Loday’s book the periodic theory is discussed much later in Chap. V.

Another topic which should be discussed is the notion of the character of a cycle [4, Sec. III.1.α]. This gives a natural way to construct cyclic cocycles. This sheds new light on the 4. talk. Finally present Theorems 2.19 and 2.21 in [11, Sec. 2.3].

Material: [11, Sec. 2.4], [3, Theorem 46], [10, Sec. 8.5]

For the algebra of smooth functions on a manifold the (continuous) Hochschild homology is canonically isomorphic to the space of differential forms and the (continuous) periodic cyclic homology is canonically isomorphic to the $\mathbb{Z}_2$–graded de Rham cohomology. This is the content of the so–called HKRC–Theorem. As a consequence Hochschild homology may be viewed as the noncommutative substitute to differential forms and periodic cyclic homology may be viewed as the noncommutative substitute for de Rham cohomology.

A proof of this important theorem can be found in [3] and in [10]. There exists a relatively new proof due to Teleman (see the references in [10]).

9./10. The noncommutative Chern character I and II.

Material: [11, Sec. 2.5], [8], [9]

For compact manifolds the Chern character is a natural transformation from $K$–cohomology to de Rham cohomology. In light of the HKRC–Theorem we should expect that the noncommutative Chern character is a natural transformation from $K$–theory to cyclic homology. This is indeed the case. Your presentation should follow the original papers by Getzler and Getzler–Szenes.

11. Further topics. There are various options for further topics (if time permits, the SS is short). One could e.g. discuss the Dixmier trace and the Hochschild character theorem [11, Sec. 3] and the Local Index Theorem of Connes and Moscovici [6].

References